

Global existence of solutions to a parabolic-elliptic chemotaxis system with critical degenerate diffusion

Elissar Nasreddine

*Institut de Mathématiques de Toulouse, Université de Toulouse,
F-31062 Toulouse cedex 9, France*

e-mail: elissar.nasreddine@math.univ-toulouse.fr

July 19, 2012

Abstract This paper is devoted to the analysis of non-negative solutions for a degenerate parabolic-elliptic Patlak-Keller-Segel system with critical nonlinear diffusion in a bounded domain with homogeneous Neumann boundary conditions. Our aim is to prove the existence of a global weak solution under a smallness condition on the mass of the initial data, there by completing previous results on finite blow-up for large masses. Under some higher regularity condition on solutions, the uniqueness of solutions is proved by using a classical duality technique.

Keywords: Chemotaxis; Keller-Segel model; Parabolic equation; Elliptic equation; Global existence; Uniqueness.

1 Introduction

Chemotaxis is the movement of biological organisms oriented towards the gradient of some substance, called the chemoattractant. The Patlak-Keller-Segel (PKS) model (see [13], [12] and [17]) has been introduced in order to explain chemotaxis cell aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cell density u , and a reaction diffusion equation for the chemoattractant concentration φ . It reads

$$(PKS) \begin{cases} \partial_t u = \operatorname{div}(\nabla u^m - u \cdot \nabla \varphi) & x \in \Omega, t > 0, \\ -\Delta \varphi = u - \langle u \rangle & x \in \Omega, t > 0, \\ \langle \varphi(t) \rangle = 0 & t > 0, \\ \partial_\nu u = \partial_\nu \varphi = 0 & x \in \partial\Omega, t > 0, \\ u(0, x) = u_0(x) & x \in \Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, ν the outward unit normal vector to the boundary $\partial\Omega$ and $m \geq 1$. An important parameter in this model is the total mass M of cells, which is formally conserved through the evolution:

$$M = \langle u \rangle = \frac{1}{|\Omega|} \int_{\Omega} u(t, x) \, dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx. \quad (2)$$

Several studies have revealed that the dynamics of (1) depend sensitively on the parameters N , m and M . More precisely, if $N = 2$ and $m = 1$, it is well-known that the solutions of (1) may blow up in finite time if M is sufficiently large (see [17, 16]) while solutions are global in time for M sufficiently small [17], see also the survey articles [4, 10].

The situation is very different when $m = 1$ and $N \neq 2$. In fact, if $N = 1$, there is global existence of solutions of (1) whatever the value of the mass of initial data M , see [8] and the references therein. If $N \geq 3$, for all $M > 0$, there are initial data u_0 with mass M for which the corresponding solutions of (1) explode in finite time (see [16]). Thus,

in dimension $N \geq 3$ and $m = 1$, the threshold phenomenon does not take place as in dimension 2, but we expect the same phenomenon when $N \geq 3$ and m is equal to the critical value $m = m_c = \frac{2(N-1)}{N}$. More precisely, we consider a more general version of (1) where the first equation of (1) is replaced by

$$\partial_t u = \operatorname{div}(\phi(u) \nabla u - u \nabla \varphi), \quad t > 0, \quad x \in \Omega,$$

and the diffusivity ϕ is a positive function in $C^1([0, \infty[)$ which does not grow too fast at infinity. In [8], the authors proved that there is a critical exponent such that, if the diffusion has a faster growth than the one given by this exponent, solutions to (1) (with $\phi(u)$ instead of mu^{m-1}) exist globally and are uniformly bounded, see also [6, 14] for $N = 2$. More precisely, the main results in [8] read as follows:

- If $\phi(u) \geq c(1+u)^p$ for all $u \geq 0$ and some $c > 0$ and $p > 1 - \frac{2}{N}$ then all solutions of (1) are global and bounded.
- If $\phi(u) \leq c(1+u)^p$ for all $u \geq 0$ and some $c > 0$ and $p < 1 - \frac{2}{N}$ then there exist initial data u_0 such that

$$\lim_{t \rightarrow T} \|u(\cdot, t)\|_\infty = \infty, \text{ for some finite } T > 0.$$

Except for $N = 2$, the critical case $m = \frac{2(N-1)}{N}$ is not covered by the analysis of [8]. Recently, Cieřlak and Laurençot in [7] show that if $\phi(u) \leq c(1+u)^{1-\frac{2}{N}}$ and $N \geq 3$, there are solutions of (1) blowing up in finite time when M exceeds an explicit threshold. In order to prove that, when $N \geq 3$ and $m = \frac{2(N-1)}{N}$, we have a threshold phenomenon similar to dimension $N = 2$ with $m = 1$, it remains to show that solutions of (1) are global when M is small enough. The goal of this paper is to show that this is indeed true, see Theorem 2.2 below.

By combining Theorem 2.2 with the blow-up result obtained in [7], we conclude that, for $N \geq 3$ and $m = \frac{2(N-1)}{N}$, there exists $0 < M_1 \leq M_2 < \infty$ such that the solutions of (1) are global if the mass M of the initial data u_0 is in $[0, M_1)$, and may explode in finite time if $M > M_2$. An important open question is whether $M_1 = M_2$ when Ω is a ball in \mathbb{R}^N and u_0 is a radially symmetric function. Notice that, in the radial case, this result is true when $N = 2$ and $m = 1$, and the threshold value of the mass for blow-up is $M_1 = M_2 = 8\pi$, see [6, 16, 15, 18]. Again, for $N = 2$ and $m = 1$, but for regular, connected and bounded domain, it has been shown that $M_1 = 4\pi = \frac{M_2}{2}$ (see [15, 16] and the references therein). Such a result does not seem to be known for $N \geq 3$ and $m = \frac{2(N-1)}{N}$.

Still, in the whole space $\Omega = \mathbb{R}^N$ when the equation for φ in (1) is replaced by the Poisson equation $\varphi = E_N * u$, with E_N being the Poisson kernel, it has been shown in [9, 5, 2, 20, 21, 3] that:

- When $N \geq 3$ and $1 \leq m < 2 - \frac{2}{N}$, this modified version of (1) has a global weak solution if $M = \|u_0\|_1$ is sufficiently small, while finite time blow-up occurs for some initial data with sufficiently large mass.
- When $N \geq 2$ and $m > 2 - \frac{2}{N}$, this modified version of (1) has a global weak solution whatever the value of M .

- When $N \geq 2$ and $m = 2 - \frac{2}{N}$, there is a threshold mass $M_c > 0$ such that solutions to this modified version of (1) exist globally if $M = \|u_0\|_1 \leq M_c$, and might blow up in finite time if $M > M_c$.

From now on, we assume that

$$N \geq 3 \quad \text{and} \quad m = \frac{2(N-1)}{N}.$$

2 Main Theorem

Throughout this paper, we deal with weak solutions of (1). Our definition of weak solutions now reads:

Definition 2.1. *Let $T \in (0; \infty]$. A pair (u, φ) of functions $u : \Omega \times [0, T) \rightarrow [0, \infty)$, $\varphi : \Omega \times [0, T) \rightarrow \mathbb{R}$ is called a weak solution of (1) in $\Omega \times [0, T)$ if*

- $u \in L^\infty((0, T); L^\infty(\Omega))$; $u^m \in L^2((0, T); H^1(\Omega))$ and $\langle u \rangle = M$.
- $\varphi \in L^2((0, T); H^1(\Omega))$ and $\langle \varphi \rangle = 0$.
- (u, φ) satisfies the equation in the sense of distributions ; i.e.,

$$\begin{aligned} - \int_0^T \int_\Omega (\nabla u^m \cdot \nabla \psi - u \nabla \varphi \cdot \nabla \psi - u \partial_t \psi) \, dx dt &= \int_\Omega u_0(x) \psi(0, x) \, dx, \\ \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx dt &= \int_0^T \int_\Omega (u - M) \psi \, dx dt, \end{aligned}$$

for any continuously differentiable function $\psi \in C^1([0, T] \times \overline{\Omega})$ with $\psi(T) = 0$ and $T > 0$.

For $\varphi \in H^1(\Omega)$ satisfying $\langle \varphi \rangle = 0$, we denote by C_s the Sobolev constant where

$$\|\nabla \varphi\|_2 \geq C_s \|\varphi\|_{2^*}, \quad \text{where } 2^* = \frac{2N}{N-2}. \quad (3)$$

The main theorem gives the existence and uniqueness of a time global weak solution to (1) which corresponds to a degenerate version of the ‘‘Nagai model’’ for the semi-linear Keller-Segel system, when $u_0 \in L^\infty(\Omega)$ and the initial data is assumed to be small.

Theorem 2.2. *Define*

$$M_* := \left(\frac{2 C_s^2}{(m-1) |\Omega|^{\frac{2}{N}}} \right)^{\frac{N}{2}}, \quad (4)$$

where C_s is the Sobolev constant in (3).

Assume that u_0 is nonnegative function in $L^\infty(\Omega)$, which satisfies

$$\|u_0\|_1 < M_*. \quad (5)$$

Then the equation (1) has a global weak solution (u, φ) in the sense of Definition 2.1. Moreover, if we assume that

$$\varphi \in L^\infty((0, T); W^{2,\infty}(\Omega)) \quad (6)$$

for all $T > 0$ then this solution is unique.

In order to prove the previous theorem, we introduce the following approximated equations

$$(KS)_\delta \begin{cases} \partial_t u_\delta &= \operatorname{div}(\nabla(u_\delta + \delta)^m - u_\delta \nabla \varphi_\delta) & x \in \Omega, t > 0, \\ -\Delta \varphi_\delta &= u_\delta - \langle u_\delta \rangle & x \in \Omega, t > 0, \\ \partial_\nu u_\delta = \partial_\nu \varphi_\delta &= 0 & x \in \partial\Omega, t > 0, \\ u_\delta(0, x) &= u_0(x) & x \in \Omega, \end{cases}$$

where $\delta \in (0, 1)$, and we show that under a smallness condition on the mass of initial data, the Liapunov function

$$L_\delta(u, \varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2} |\nabla \varphi_\delta|^2 - u_\delta \varphi_\delta) \, dx,$$

yields the L^m bound of $u_\delta(t)$ independent of δ . Then using Gagliardo-Nirenberg and Poincaré inequalities, we obtain for $p > m$, the L^p bound for $u_\delta(t)$ independent of δ . As a consequence of Sobolev embedding theorem, we improve the regularity of φ_δ . And thus, under the same assumptions on the initial data, Moser's iteration technique yields the uniform bound of u_δ . Then, thanks to the local well-posedness result [8, Theorem 3.1] we obtain the existence of a global solution of $(KS)_\delta$. The existence of solutions stated in Theorem 2.2 is then proved using a compactness method; for that purpose we show an additional estimate on $\partial_t u_\delta^m$ which, together with the already derived estimates, guarantees the compactness in space and time of the family $(u_\delta)_{\delta \in (0,1)}$. Finally, in the presence of nonlinear diffusion and under some additional regularity assumption on φ_δ , we prove the uniqueness using a classical duality technique.

3 Approximated Equations

The first equation of (1) is a quasilinear parabolic equation of degenerate type. Therefore, we cannot expect the system (1) to have a classical solution at the point where u vanishes. In order to prove Theorem 2.2, we use a compactness method and introduce the following approximated equations of (KS):

$$(KS)_\delta \begin{cases} \partial_t u_\delta &= \operatorname{div}(\nabla(u_\delta + \delta)^m - u_\delta \nabla \varphi_\delta) & x \in \Omega, t > 0, \\ -\Delta \varphi_\delta &= u_\delta - \langle u_\delta \rangle & x \in \Omega, t > 0, \\ \partial_\nu u_\delta = \partial_\nu \varphi_\delta &= 0 & x \in \partial\Omega, t > 0, \\ u_\delta(0, x) &= u_0(x) & x \in \Omega, \end{cases} \quad (7)$$

where $\delta \in (0, 1)$.

The main purpose of this section is to construct the time global strong solution of (7).

3.1 Existence of global strong solution of $(KS)_\delta$

Theorem 3.1. *For $\delta \in (0, 1)$ and $T > 0$, we consider an initial condition $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and such that $\|u_0\|_1 < M_*$ where M_* is defined in (4). Then $(KS)_\delta$ has a global strong solution $(u_\delta, \varphi_\delta)$ which is bounded in $L^\infty((0, T) \times \Omega)$ for all $T > 0$ uniformly with respect to $\delta \in (0, 1)$.*

The starting point of the proof of Theorem 3.1 is the following local well-posedness result [8, Theorem 1.3]:

Lemma 3.2. *Let the same assumptions as that in Theorem 3.1 hold. There exists a maximal existence time $T_{max}^\delta \in (0, \infty]$ and a unique solution $(u_\delta, \varphi_\delta)$ of $(KS)_\delta$ in $[0, T_{max}^\delta) \times \Omega$. Moreover,*

$$\text{if } T_{max}^\delta < \infty \text{ then } \lim_{t \rightarrow T_{max}^\delta} \|u_\delta(t, \cdot)\|_\infty = \infty.$$

In addition $\langle u_\delta(t) \rangle = \langle u_0 \rangle = M$ for all $t \in [0, T_{max}^\delta)$.

To prove Theorem 3.1 we need to prove some lemmas which control L^m norm, L^p norm and L^∞ norm of the solution u_δ of (7).

3.2 L^p -estimates, $1 \leq p \leq \infty$.

Our goal is to show that if $\|u_0\|_1$ is small enough then all solutions are global in time and uniformly bounded.

Let us first prove the L^m bound for u_δ .

Lemma 3.3. *Let the same assumptions as that in Theorem 3.1 hold and $(u_\delta, \varphi_\delta)$ be the nonnegative maximal solution of $(KS)_\delta$. Then, u_δ satisfies the following estimate*

$$\|u_\delta(t)\|_m \leq C_0, \text{ for all } t \in [0, T_{max}^\delta)$$

and $\|u_\delta(t)\|_1 = \|u_0\|_1$ where C_0 is a constant independent of T_{max}^δ and δ .

Proof. In this proof, the solution to equation (7) should be denoted by $(u_\delta, \varphi_\delta)$ but for simplicity we drop the index.

Let us define the functional L_δ by

$$L_\delta(u, \varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2} |\nabla \varphi|^2 - u \varphi) dx,$$

where

$$b_\delta(u) := \int_1^u \int_1^z \frac{m(\sigma + \delta)^{m-1}}{\sigma} d\sigma dz,$$

such that $b_\delta(1) = b'_\delta(1) = 0$ and $b(u) \geq 0$. According to [11] it is a Liapunov functional for $(KS)_\delta$. Indeed,

$$\begin{aligned} \frac{d}{dt} L_\delta(u(t), \varphi(t)) &= \int_\Omega b'_\delta(u) \partial_t u dx - \int_\Omega \Delta \varphi \partial_t \varphi dx - \int_\Omega \partial_t u \varphi dx - \int_\Omega u \partial_t \varphi dx \\ &= \int_\Omega \partial_t u (b'_\delta(u) - \varphi) dx - \int_\Omega (\Delta \varphi + u) \partial_t \varphi dx \\ &= \int_\Omega \operatorname{div} (m (u + \delta)^{m-1} \nabla u - u \nabla \varphi) (b'_\delta(u) - \varphi) dx - \int_\Omega \langle u(t) \rangle \partial_t \varphi dx \\ &= - \int_\Omega (m (u + \delta)^{m-1} \nabla u - u \nabla \varphi) (b''_\delta(u) \nabla u - \nabla \varphi) dx - M \frac{d}{dt} \int_\Omega \varphi dx \\ &= - \int_\Omega u (b''_\delta(u) \nabla u - \nabla \varphi)^2 dx \\ &\leq 0. \end{aligned}$$

Then, we can conclude that for all $t \in [0, T_{max}^\delta)$ we have $L_\delta(u(t), \varphi(t)) \leq L_\delta(u_0, \varphi_0)$. Using Sobolev inequality (3), Hölder inequality, and Young inequality we obtain

$$\int_\Omega u \varphi dx \leq \|\varphi\|_{2^*} \|u\|_{\frac{2N}{N+2}} \leq C_s^{-1} \|\nabla \varphi\|_2 \|u\|_{\frac{2N}{N+2}} \leq \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{C_s^{-2}}{2} \|u\|_{\frac{2N}{N+2}}^2.$$

Since $\frac{2}{N+2} < m$, and using interpolation inequality we get,

$$\|u\|_{\frac{2N}{N+2}} \leq \|u\|_1^{\frac{1}{N}} \|u\|_m^{\frac{N-1}{N}} \leq M^{\frac{1}{N}} |\Omega|^{\frac{1}{N}} \|u\|_m^{\frac{m}{2}}.$$

Then,

$$\int_{\Omega} u \varphi \, dx \leq \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \|u\|_m^m.$$

Substituting this into the Liapunov functional, we find:

$$\begin{aligned} L_{\delta}(u, \varphi) &\geq \int_{\Omega} (b_{\delta}(u) + \frac{1}{2} |\nabla \varphi|^2) \, dx - \frac{1}{2} \|\nabla \varphi\|_2^2 - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \|u\|_m^m \\ &\geq \int_{\Omega} b_{\delta}(u) \, dx - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \|u\|_m^m. \end{aligned}$$

We next observe that:

$$\begin{aligned} b_{\delta}(u) &= m \int_1^u \int_1^z \frac{(\delta + s)^{m-1}}{s} \, ds dz \geq m \int_1^u \int_1^z s^{m-2} \, ds dz \\ &\geq \frac{u^m}{m-1} - \frac{m}{m-1} u + 1 \geq \frac{u^m}{m-1} - \frac{m}{m-1} u. \end{aligned}$$

Then:

$$\begin{aligned} L_{\delta}(u, \varphi) &\geq \frac{1}{m-1} \|u\|_m^m - \frac{C_s^{-2}}{2} |\Omega|^{\frac{2}{N}} M^{\frac{2}{N}} \|u\|_m^m - \frac{m}{m-1} M |\Omega| \\ &= \left(\frac{1}{m-1} - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \right) \|u\|_m^m - \frac{m}{m-1} M |\Omega|. \end{aligned}$$

Let us define ω_M by

$$\omega_M := \frac{1}{m-1} - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} = \frac{|\Omega|^{\frac{2}{N}}}{2 C_s^2} (M_*^{\frac{2}{N}} - M^{\frac{2}{N}}).$$

Since $M = \|u_0\|_1 < M_*$, then ω_M is positive. Finally we get,

$$L_{\delta}(u_0, \varphi_0) + \frac{m}{m-1} M |\Omega| \geq L_{\delta}(u(t), \varphi(t)) + \frac{m}{m-1} M |\Omega| \geq \omega_M \|u(t)\|_m^m \text{ for } t \in [0, T_{max}^{\delta}).$$

In addition, we can see that $L_{\delta}(u_0, \varphi_0) \leq C$ where C is independent of $\delta \in (0, 1)$. In fact,

$$L_{\delta}(u_0, \varphi_0) = \int_{\Omega} (b_{\delta}(u_0) + \frac{1}{2} |\nabla \varphi_0|^2 - u_0 \varphi_0) \, dx,$$

and, since $(\delta + s)^{m-1} \leq \delta^{m-1} + s^{m-1} \leq 1 + s^{m-1}$ we obtain

$$\begin{aligned} b_{\delta}(u_0) &= m \int_1^{u_0} \int_1^z \frac{(\delta + s)^{m-1}}{s} \, ds dz \leq m \int_1^{u_0} \int_1^z \frac{1 + s^{m-1}}{s} \, ds dz \\ &\leq m(u_0 \ln u_0 - u_0 + 1) + \frac{m}{m-1} \left(\frac{u_0^m}{m} - u_0 + 1 \right). \end{aligned}$$

Using Young inequality we get

$$L_{\delta}(u_0, \varphi_0) \leq m \|u_0\|_2^2 + m |\Omega| + \frac{\|u_0\|_m^m}{m-1} + \frac{m |\Omega|}{m-1} + \frac{1}{2} \|\nabla \varphi_0\|_2^2 + \frac{1}{2} \|u_0\|_2^2 + \frac{1}{2} \|\varphi_0\|_2^2.$$

since $u_0 \in L^{\infty}(\Omega)$ and $\varphi_0 \in H^1(\Omega)$ we get $L_{\delta}(u_0, \varphi_0) \leq C$ where C is independent of δ and the proof of the lemma is complete. \square

Thanks to Lemma 3.3, let us now show that for all $p > m$ the L^p bound for u_δ .

Lemma 3.4. *Let the same assumptions as that in Theorem 3.1 hold. Then for all $T > 0$ and all $p \in (1, \infty)$ there exists $C(p, T)$ independent on δ such that, for all $t \in [0, T_{\max}^\delta) \cap [0, T]$, the solution $(u_\delta, \varphi_\delta)$ to $(KS)_\delta$ satisfies*

$$\|u_\delta(t)\|_p \leq C(p, T), \quad (8)$$

and

$$\int_0^t \int_\Omega (\delta + u_\delta)^{m-1} u_\delta^{p-2} |\nabla u_\delta|^2 dx ds \leq C(p, T). \quad (9)$$

To prove the previous lemma we need the following preliminary result [20].

Lemma 3.5. *Consider $0 < q_1 < q_2 \leq 2^*$. There is C_1 depending only on N such that*

$$\|u\|_{q_2} \leq C_1^\theta \|u\|_{H^1(\Omega)}^\theta \|u\|_{q_1}^{1-\theta}, \text{ for } u \in H^1(\Omega), \quad (10)$$

with

$$\theta = \frac{2N(q_2 - q_1)}{q_2[(N+2)q_1 + 2N(1 - q_1)]} \in [0, 1].$$

Proof. For $u \in H^1(\Omega)$ we have by Sobolev inequality

$$\|u\|_{2^*} \leq C_N \|u\|_{H^1}. \quad (11)$$

By interpolation inequality we have for $0 < q_1 < q_2 \leq 2^*$

$$\|u\|_{q_2} \leq \|u\|_{2^*}^\theta \|u\|_{q_1}^{1-\theta}, \quad (12)$$

where $\frac{1}{q_2} = \frac{\theta(N-2)}{2N} + \frac{1-\theta}{q_1}$. Hence, substitute (11) into (12) and the lemma is proved. \square

Now, we recall the following generalized Poincaré inequality.

Lemma 3.6. *For $u \in H^1(\Omega)$ we have for $0 < q_1 \leq 1$ the following inequality*

$$\|u\|_{H^1}^2 \leq C_2(q_1) (\|\nabla u\|_2^2 + \|u\|_{q_1}^2),$$

where C_2 depends only on Ω and q_1 .

Now using the last two lemmas, let us prove Lemma 3.4.

Proof. In this proof, the solution to equation (7) should be denoted by $(u_\delta, \varphi_\delta)$ but for simplicity we drop the index.

We choose $p > 1$, $K \geq 0$ and we multiply the first equation in (7) by $(u - K)_+^{p-1}$ and

integrate by parts using the boundary conditions for u and φ to see that

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|(u - K)_+\|_p^p &= -m(p-1) \int_{\Omega} (\delta + u)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ (p-1) \int_{\Omega} u \nabla \varphi (u - K)_+^{p-2} \cdot \nabla u dx \\
&= -m(p-1) \int_{\Omega} (\delta + u - K + K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ (p-1) \int_{\Omega} (u - K + K) \nabla \varphi \cdot (u - K)_+^{p-2} \nabla u dx \\
&\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ (p-1) \int_{\Omega} (u - K)_+^{p-1} \nabla \varphi \cdot \nabla u dx + (p-1)K \int_{\Omega} \nabla \varphi (u - K)_+^{p-2} \cdot \nabla u dx \\
&\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&- \frac{p-1}{p} \int_{\Omega} (u - K)_+^p \Delta \varphi dx - K \int_{\Omega} (u - K)_+^{p-1} \Delta \varphi dx \\
&\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx + (I),
\end{aligned}$$

where, thanks to the second equation in (7),

$$\begin{aligned}
(I) &= \frac{p-1}{p} \int_{\Omega} (u - K)_+^p (u - M) dx + K \int_{\Omega} (u - K)_+^{p-1} (u - M) dx \\
&= \frac{p-1}{p} \|(u - K)_+\|_{p+1}^{p+1} + \frac{p-1}{p} (K - M) \|(u - K)_+\|_p^p \\
&+ K \|(u - K)_+\|_p^p + K(K - M) \|(u - K)_+\|_{p-1}^{p-1} \\
&\leq K^2 \|(u - K)_+\|_{p-1}^{p-1} + 2K \|(u - K)_+\|_p^p + \|(u - K)_+\|_{p+1}^{p+1}.
\end{aligned}$$

Since for $a > 0$ and $b > 0$ we have $a^{p-1}b \leq a^{p+1} + b^{\frac{p+1}{2}}$ and $a^p b \leq a^{p+1} + b^{p+1}$ then,

$$(I) \leq 3 \|(u - K)_+\|_{p+1}^{p+1} + (2K)^{p+1} + K^{p+1}, \quad (13)$$

and we get

$$\begin{aligned}
\frac{d}{dt} \|(u - K)_+\|_p^p &\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ 3p \|(u - K)_+\|_{p+1}^{p+1} + C_p K^{p+1},
\end{aligned} \quad (14)$$

for all $t \in [0, T_{\max}^\delta)$.

The term $\|(u - K)_+\|_{p+1}^{p+1}$ can be estimated with the help of Lemma 3.5 and Lemma 3.6.

Assuming now that $p > 2$ we remark that $0 < \frac{2}{p+m-1} \leq 1$ and $1 < \frac{2(p+1)}{p+m-1} = \frac{2N}{N-2} \frac{1+p}{1+\frac{Np}{N-2}} \leq \frac{2N}{N-2}$, then thanks to Lemma 3.5 and Lemma 3.6 we obtain

$$\begin{aligned}
\|(u - K)_+^{\frac{p+m-1}{2}}\|_{\frac{2(p+1)}{p+m-1}}^{\frac{2(p+1)}{p+m-1}} &\leq C(p) \left(\|\nabla(u - K)_+^{\frac{p+m-1}{2}}\|_2^{\frac{2(p+1)}{p+m-1} \theta} \|(u - K)_+^{\frac{p+m-1}{2}}\|_{\frac{2(p+1)}{p+m-1}}^{\frac{2(p+1)}{p+m-1} (1-\theta)} \right. \\
&\quad \left. + \|(u - K)_+^{\frac{p+m-1}{2}}\|_{\frac{2(p+1)}{p+m-1}}^{\frac{2(p+1)}{p+m-1}} \right),
\end{aligned} \quad (15)$$

where

$$\theta = \frac{p+m-1}{p+1} \in (0, 1). \quad (16)$$

Since

$$\|(u-K)_+\|^{\frac{p+m-1}{2}}_{\frac{2(p+1)}{p+m-1}} = \int_{\Omega} (u-K)_+^{p+1} dx = \|(u-K)_+\|_{p+1}^{p+1}, \quad (17)$$

$$\|(u-K)_+\|^{\frac{p+m-1}{2}}_{\frac{2(p+1)}{p+m-1}}^{(1-\theta)} = \left(\int_{\Omega} (u-K)_+ dx \right)^{(p+1)(1-\theta)} = \|(u-K)_+\|_1^{\frac{2}{N}}, \quad (18)$$

and by Lemma 3.3

$$\|(u-K)_+\|_1 = \int_{u \geq K} (u-K) dx \leq \frac{1}{K^{m-1}} \int_{u \geq K} K^{m-1} u dx \leq \frac{\|u\|_m^m}{K^{m-1}} \leq \frac{C_0^m}{K^{m-1}}, \quad (19)$$

we substitute (17), (18) and (19) into (15) and obtain

$$\|(u-K)_+\|_{p+1}^{p+1} \leq C_3(p) \left\{ \|\nabla(u-K)_+\|^{\frac{m+p-1}{2}}_2^2 K^{\frac{-2(m-1)}{N}} + K^{-(m-1)(p+1)} \right\}. \quad (20)$$

We may choose $K = K_*$ large enough such that

$$3 p C_3(p) K_*^{\frac{-2(m-1)}{N}} \leq \frac{2 p (p-1) m}{(m+p-1)^2},$$

Hence

$$\frac{d}{dt} \|(u-K_*)_+\|_p^p \leq C(p) K_*^{p+1},$$

so that

$$\|(u(t)-K_*)_+\|_p^p \leq C(p) t + \|u_0\|_p^p, \text{ for } t \in [0, T_{\max}^\delta].$$

As

$$\begin{aligned} \int_{\Omega} |u|^p dx &\leq \int_{u < 2K_*} (2K_*)^{p-1} |u| dx + \int_{u \geq 2K_*} |u - K_* + K_*|^p dx \\ &\leq (2K_*)^{p-1} M + \int_{u \geq 2K_*} (2|u - K_*|)^p dx, \\ &\leq (2K_*)^{p-1} M + 2^p \|(u-K_*)_+\|_p^p, \end{aligned}$$

the previous inequality warrants that

$$\|u(t)\|_p \leq C(p, T), \quad t \in [0, T_{\max}) \cap [0, T], \quad (21)$$

where $C(p, T)$ is a constant independent of δ .

We next take $K = 0$ in (14), integrate with respect to time and use (8) to obtain (9). \square

Thanks to Lemma 3.4, we can improve the regularity of φ_δ .

Lemma 3.7. *Let the same assumptions as that in Theorem 3.1 hold, the solution φ_δ satisfies*

$$\|\nabla \varphi_\delta(t)\|_\infty \leq L(T), \quad t \in [0, T_{\max}^\delta) \cap [0, T]$$

where $T > 0$ and L is a positive constant independent of δ .

Proof. Using standard elliptic regularity estimates for φ_δ , we infer from Lemma 3.4 that given $T > 0$, and $p \in (1, \infty)$, there is $C(p, T)$ such that

$$\|\varphi_\delta(t)\|_{W^{2,p}} \leq C(p) \|u_\delta(t)\|_p \leq C(p, T), \text{ for } t \in [0, T_{\max}) \cap [0, T].$$

Lemma 3.7 then readily follows from Sobolev embedding theorem upon choosing $p > N$. \square

Lemma 3.8. *Let $N \geq 3$, $r \geq 4$, $u \in L^{\frac{r}{4}}(\Omega)$, and $u^{\frac{r+m-1}{2}} \in H^1(\Omega)$. Then it holds that*

$$\|u\|_r \leq C_1^{\frac{2\theta}{r+m-1}} \|u\|_{\frac{r}{4}}^{1-\theta} \|u^{\frac{r+m-1}{2}}\|_{H^1}^{\frac{2\theta}{r+m-1}} \quad (22)$$

with

$$\theta = \frac{3N(r+m-1)}{(3N+2)r+4N(m-1)} \in (0, 1). \quad (23)$$

Proof. For $r \geq 4$, we can see that

$$\|u\|_r = \left(\int_{\Omega} (u^{\frac{r+m-1}{2}})^{\frac{2r}{r+m-1}} dx \right)^{\frac{1}{r}} = \|u^{\frac{r+m-1}{2}}\|_{\frac{2r}{r+m-1}},$$

and

$$\frac{r}{2(r+m-1)} < 1 < \frac{2r}{r+m-1} < 2 < \frac{2N}{N-2}.$$

By Lemma 3.5,

$$\|u\|_r = \|u^{\frac{r+m-1}{2}}\|_{\frac{2r}{r+m-1}}^{\frac{2}{r+m-1}} \leq \left(C_1^\theta \|u^{\frac{r+m-1}{2}}\|_{H^1(\Omega)}^\theta \|u^{\frac{r+m-1}{2}}\|_{\frac{r}{2(r+m-1)}}^{1-\theta} \right)^{\frac{2}{r+m-1}}$$

and

$$\begin{aligned} \theta &= \frac{2N \left(\frac{2r}{r+m-1} - \frac{r}{2(r+m-1)} \right)}{\frac{2r}{r+m-1} \left(2N \left(1 - \frac{r}{2(r+m-1)} \right) + (N+2) \frac{r}{2(r+m-1)} \right)} \\ &= \frac{3N(r+m-1)}{(3N+2)r+4N(m-1)} \in (0, 1). \end{aligned}$$

In addition, we have

$$\|u^{\frac{r+m-1}{2}}\|_{\frac{r}{2(r+m-1)}} = \left(\int_{\Omega} |u|^{\frac{r+m-1}{2} \frac{r}{2(r+m-1)}} dx \right)^{\frac{2(r+m-1)}{r}} = \|u\|_{\frac{r}{4}}^{\frac{r+m-1}{2}},$$

and we obtain (22). \square

We are now in a position to prove the uniform $L^\infty(\Omega)$ bound for u_δ .

Lemma 3.9. *Let the same assumptions as that in Theorem 3.1 hold, and $(u_\delta, \varphi_\delta)$ be the nonnegative maximal solution of (7). For all $T > 0$, there is $C_\infty(T)$ such that*

$$\|u_\delta(t)\|_\infty \leq C_\infty(T), \text{ for all } t \in [0, T_{\max}^\delta) \cap [0, T],$$

where $C_\infty(T)$ is a positive constant independent on δ .

Proof. In this proof we omit the index δ , and we employ Moser's iteration technique developed in [1, 21] to show the uniform norm bound for u .

We multiply the first equation in (7) by u^{r-1} , where $r \geq 4$, and integrate it over Ω . Then, we have

$$\begin{aligned} \frac{d}{dt} \frac{\|u\|_r^r}{r} &= - \int_{\Omega} (\nabla(u + \delta)^m - u \nabla \varphi) \cdot \nabla u^{r-1} dx \\ &= -m(r-1) \int_{\Omega} (u + \delta)^{m-1} u^{r-2} |\nabla u|^2 dx + (r-1) \int_{\Omega} u^{r-1} \nabla \varphi \cdot \nabla u dx \\ &\leq -m(r-1) \int_{\Omega} u^{m+r-3} |\nabla u|^2 dx + (r-1) \int_{\Omega} u^{r-1} \nabla \varphi \cdot \nabla u dx. \end{aligned}$$

By Young's inequality and Lemma 3.7,

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|u\|_r^r &\leq \frac{-4m(r-1)}{(r+m-1)^2} \int_{\Omega} |\nabla u^{\frac{r+m-1}{2}}|^2 dx + \frac{2(r-1) \|\nabla \varphi\|_{\infty}}{(r+m-1)} \int_{\Omega} u^{\frac{r-m+1}{2}} |\nabla u^{\frac{r+m-1}{2}}| dx \\ &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + \frac{r-1}{2m} \|\nabla \varphi\|_{\infty}^2 \int_{\Omega} u^{r-m+1} dx \\ &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C(T) r \int_{\Omega} u^{r-m+1} dx. \end{aligned}$$

Using Hölder and Young inequalities and Lemma 3.3 we obtain

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|u\|_r^r &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + r C(T) \|u\|_1^{\frac{m-1}{r-1}} \|u\|_r^{\frac{r(r-m)}{r-1}} \\ &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C^r + r^2 \|u\|_r^r, \end{aligned} \quad (24)$$

where we have used that $r^{\frac{r-1}{r-m}} \leq r^2$ for $r \geq 4$.

By Lemma 3.8, we have for $r \geq 4$

$$\|u\|_r^r \leq C_1^{\frac{2}{r+m-1}} \|u\|_{\frac{r}{4}}^{r(1-\theta)} \|u^{\frac{r+m-1}{2}}\|_{H^1}^{\frac{2}{r+m-1}}, \quad (25)$$

where

$$\theta = \frac{3N(r+m-1)}{(3N+2)r+4N(m-1)} < 1.$$

Therefore, Young inequality and (25) yield that

$$\begin{aligned} 2r^2 \|u\|_r^r &\leq 2r^2 C_1^{\frac{2}{r+m-1}} \|u\|_{\frac{r}{4}}^{r(1-\theta)} \|u^{\frac{r+m-1}{2}}\|_{H^1}^{\frac{2}{r+m-1}} \\ &\leq \frac{\theta r}{r+m-1} \frac{m(r-1)}{(r+m-1)^2} \frac{r+m-1}{\theta r C_2(1)} \|u^{\frac{r+m-1}{2}}\|_{H^1}^2 \\ &\quad + \frac{r+m-1-\theta r}{r+m-1} \left(C_2(1) \frac{\theta(r+m-1)r}{m(r-1)} \right)^{\frac{\theta r}{r(1-\theta)+m-1}} \\ &\quad \times (2r^2)^{\frac{(r+m-1)}{r(1-\theta)+m-1}} C_1^{\frac{2}{r(1-\theta)+m-1}} \|u\|_{\frac{r}{4}}^{(1-\theta)r \frac{(r+m-1)}{r(1-\theta)+m-1}}, \end{aligned}$$

where $C_2(1)$ is the Poincaré constant defined in Lemma 3.6. Then we obtain

$$\begin{aligned} 2r^2 \|u\|_r^r &\leq \frac{m(r-1)}{C_2(1)(r+m-1)^2} \|u^{\frac{r+m-1}{2}}\|_{H^1}^2 \\ &\quad + C_1^{\frac{\theta r}{r(1-\theta)+m-1}} 2^{\frac{(r+m-1)}{r(1-\theta)+m-1}} r^{\frac{2(r+m-1)+\theta r}{r(1-\theta)+m-1}} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}. \end{aligned}$$

Now, since $N > 2$, which gives $4N \geq 3N + 2$, we find the following upper bound for θ

$$\theta \leq \frac{3N}{3N+2} \quad (26)$$

In addition,

$$\frac{\theta r}{r(1-\theta) + m - 1} \leq \frac{\theta}{1-\theta} = -1 + \frac{1}{1-\theta} \leq \frac{3N}{2}, \quad (27)$$

$$\frac{r+m-1}{r(1-\theta) + m - 1} \leq \frac{r+m-1}{(1-\theta)(r+m-1)} \leq \frac{1}{1-\theta} \leq \frac{3N+2}{2}, \quad (28)$$

and

$$\frac{2(r+m-1) + \theta r}{r(1-\theta) + m - 1} \leq \frac{2+\theta}{1-\theta} \leq 9N+4. \quad (29)$$

As $C_1 \geq 1$ and $r \geq 1$, we get

$$2 r^2 \|u\|_r^r \leq \frac{m(r-1)}{C_2(1)(r+m-1)^2} \|u^{\frac{r+m-1}{2}}\|_{H^1}^2 + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}. \quad (30)$$

Using Lemma 3.6 we have

$$\|u^{\frac{r+m-1}{2}}\|_{H^1}^2 \leq C_2(1) \left(\|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + \|u^{\frac{r+m-1}{2}}\|_1^2 \right). \quad (31)$$

Using Hölder inequality, Young inequality and Lemma 3.3, we get

$$\|u^{\frac{r+m-1}{2}}\|_1^2 = \|u\|_{\frac{r+m-1}{2}}^{r+m-1} \leq \|u\|_r^{\frac{r+m-3}{r-1}} \|u\|_1^{\frac{r-m+1}{r-1}} \leq \|u\|_r^{\frac{r+m-3}{r-1}} \|u_0\|_1^{\frac{r-m+1}{r-1}},$$

then,

$$\begin{aligned} \frac{m(r-1)}{(r+m-1)^2} \|u^{\frac{r+m-1}{2}}\|_1^2 &\leq (r-1)^{\frac{r-1}{r+m-3}} \frac{r+m-3}{r-1} \|u\|_r^r \\ &\quad + \frac{2-m}{r-1} \left(\frac{m}{(r+m-1)^2} \|u_0\|_1^{\frac{r+m-1}{r-1}} \right)^{\frac{r-1}{2-m}} \\ &\leq r^2 \|u\|_r^r + \left(\frac{m}{(r+m-1)^2} \|u_0\|_1^{\frac{r+m-1}{r-1}} \right)^{\frac{r-1}{2-m}} \\ &\leq r^2 \|u\|_r^r + C_4^r. \end{aligned} \quad (32)$$

Now substituting (32) and (31) into (30) we get

$$\begin{aligned} 2 r^2 \|u\|_r^r &\leq \frac{m(r-1)}{(r+m-1)^2} \left(\|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + \|u^{\frac{r+m-1}{2}}\|_1^2 \right) + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}} \\ &\leq \frac{m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + r^2 \|u\|_r^r + C_4^r + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}, \end{aligned}$$

hence

$$r^2 \|u\|_r^r \leq \frac{m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C_4^r + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{r(1-\theta)(r+m-1)}{r(1-\theta)+m-1}}.$$

We apply Young inequality again to the last term of the above inequality. It is easy to see that

$$\frac{2}{3N+2} \leq 1-\theta \leq \frac{(1-\theta)(r+m-1)}{r(1-\theta)+m-1} = \frac{(1-\theta)r + (1-\theta)(m-1)}{r(1-\theta)+m-1} < 1,$$

so that

$$r^2 \|u\|_r^r \leq \frac{m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C_4^r + 1 + (C r^{9N+4})^{3N+1} \|u\|_{\frac{r}{4}}^r, \quad (33)$$

for any $r \in [4, \infty)$.

Substituting (33) into (24) we end up with

$$\frac{d}{dt} \|u\|_r^r \leq r C_4^r + r + r (C r^{9N+4})^{3N+1} \|u\|_{\frac{r}{4}}^r \leq C_5^r + C r^\alpha \|u\|_{\frac{r}{4}}^r, \quad (34)$$

for any $r \in [4, \infty)$, where $\alpha = (9N+4)(3N+1) + 1$. After integrating (34) from 0 to t , we obtain the L^r estimate for u as follows:

$$\sup_{0 < t < T} \|u(t)\|_r^r \leq \|u_0\|_r^r + T C_5^r + C r^\alpha T \sup_{0 < t < T} \|u(t)\|_{\frac{r}{4}}^r. \quad (35)$$

Since

$$\|u_0\|_r \leq \|u_0\|_\infty^{\frac{r-1}{r}} \|u_0\|_1^{\frac{1}{r}} \leq C_6,$$

then

$$\sup_{0 < t < T} \|u(t)\|_r^r \leq C_7(T) r^\alpha \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{\frac{r}{4}} \right\}^r, \quad (36)$$

and we obtain for $r \geq 4$

$$\sup_{0 < t < T} \|u(t)\|_r \leq C_7(T)^{\frac{1}{r}} r^{\frac{\alpha}{r}} \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{\frac{r}{4}} \right\}. \quad (37)$$

We are now in a position to derive the claimed L^∞ estimate. To this end, we set

$$\alpha_p := \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{4^p} \right\}$$

for $p \geq 0$. Then we take $r = 4^p$ with $p \geq 0$ in (37) which reads

$$\begin{aligned} \alpha_p &\leq 4^{\frac{\alpha p}{4^p}} C_7(T)^{\frac{1}{4^p}} \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{4^{p-1}} \right\}, \\ &\leq 4^{\frac{\alpha}{2^p}} C_7(T)^{\frac{1}{4^p}} \alpha_{p-1} \end{aligned}$$

since $p \leq 2^p$ for $p \geq 1$. Arguing by induction we conclude that

$$\alpha_p \leq 4^{\alpha \sum_{k=1}^p 2^{-k}} C_7(T)^{\sum_{k=1}^p 4^{-k}} \alpha_0.$$

Then by using Lemma 3.3 we get

$$\sup_{0 < t < T} \|u(t)\|_{4^p} \leq 4^\alpha C_7(T) \alpha_0 \leq C_8(T).$$

Consequently, by letting p tend to ∞ , we see that $u \in L^\infty((0, T) \times \Omega)$ and

$$\sup_{0 < t < T} \|u(t)\|_\infty \leq C_8(T). \quad (38)$$

Since the right hand side is independent of δ , we have proved the lemma. \square

Lemma 3.10. *Let the same assumptions as that in Theorem 3.1 hold, and $(u_\delta, \varphi_\delta)$ be the solution to (7). Then for all $T > 0$ there is $C_9(T)$ such that the solution u_δ satisfies the following derivation estimate*

$$\int_0^T \|\partial_t u_\delta^m\|_{(W^{1,N+1})'} dt \leq C_9(T).$$

Proof. Consider $\psi \in W^{1,N+1}(\Omega)$ and $t \in (0, T)$, we have

$$\begin{aligned} & \left| \int_\Omega m u_\delta^{m-1}(t) \partial_t u_\delta(t) \psi dx \right| \\ &= m \left| \int_\Omega \nabla(u_\delta^{m-1} \psi) \cdot (\nabla u_\delta^m - u_\delta \nabla \varphi_\delta) dx \right| \\ &= m \left| \int_\Omega (u_\delta^{m-1} \nabla \psi + \psi \nabla u_\delta^{m-1}) \cdot (\nabla u_\delta^m - u_\delta \nabla \varphi_\delta) dx \right| \\ &\leq m \int_\Omega [u_\delta^{m-1} |\nabla u_\delta^m| |\nabla \psi| + u_\delta^m |\nabla \psi| |\nabla \varphi_\delta| \\ &\quad + |\psi| m(m-1) u_\delta^{2m-3} |\nabla u_\delta|^2 + |\psi|(m-1) u_\delta^{m-1} |\nabla u_\delta| |\nabla \varphi_\delta|] dx \\ &\leq m \left[\|u_\delta\|_\infty^{m-1} \|\nabla u_\delta^m\|_2 \|\nabla \psi\|_2 + \|\nabla \psi\|_2 \|u_\delta\|_\infty^m \|\nabla \varphi_\delta\|_\infty |\Omega|^{\frac{1}{2}} \right. \\ &\quad \left. + \|\psi\|_\infty \frac{4m(m-1)}{(2m-1)^2} \|\nabla u_\delta^{m-\frac{1}{2}}\|_2^2 + \|\psi\|_2 \frac{m-1}{m} \|\nabla u_\delta^m\|_2 \|\nabla \varphi_\delta\|_\infty \right]. \end{aligned}$$

Using Lemma 3.8, Lemma 3.9, and the embedding of $W^{1,N+1}(\Omega)$ in $L^\infty(\Omega)$, we end up with

$$|\langle \partial_t u_\delta^m(t), \psi \rangle| \leq C(T) \left(\|\nabla u_\delta(t)^m\|_2 + \|\nabla u_\delta^{m-\frac{1}{2}}(t)\|_2^2 + 1 \right) \|\psi\|_{W^{1,N+1}},$$

and a duality argument gives

$$\|\partial_t u_\delta^m(t)\|_{(W^{1,N+1})'} \leq C(T) \left(\|\nabla u_\delta^m(t)\|_2 + \|\nabla u_\delta^{m-\frac{1}{2}}(t)\|_2^2 + 1 \right).$$

Integrating the above inequality over $(0, T)$ and using Lemma 3.4 with $p = 2$ and $p = m$ give Lemma 3.10. \square

4 Proof of Theorem 2.2

4.1 Existence

In this section, we assume that u_0 is a nonnegative function in $L^\infty(\Omega)$ satisfying (5). For $\delta \in (0, 1)$, $(u_\delta, \varphi_\delta)$ denotes the solution to $(KS)_\delta$ constructed in Section 3. To prove existence of a weak solution, we use a compactness method. For that purpose, we first study the compactness properties of $(u_\delta, \varphi_\delta)_\delta$.

Lemma 4.1. *There are functions u and φ and a sequence $(\delta_n)_{n \geq 1}$, $\delta_n \rightarrow 0$, such that, for all $T > 0$ and $p \in (1, \infty)$,*

$$u_{\delta_n} \longrightarrow u, \text{ in } L^p((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0, \quad (39)$$

$$\varphi_{\delta_n} \longrightarrow \varphi, \text{ in } L^p((0, T); W^{2,p}(\Omega)) \text{ as } \delta_n \rightarrow 0. \quad (40)$$

In addition, $u \in L^\infty((0, T) \times \Omega)$ for all $T > 0$ and is nonnegative.

Proof. Thanks to Lemma 3.4 and Lemma 3.9, $(u_\delta^m)_\delta$ is bounded in $L^2((0, T); H^1(\Omega))$ while $(\partial_t u_\delta^m)_\delta$ is bounded in $L^1((0, T); (W^{1, N+1})'(\Omega))$ by Lemma 3.10.

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and $L^2(\Omega)$ is continuously embedded in $(W^{1, N+1})'(\Omega)$, it follows from [19, corollary 4] that (u_δ^m) is compact in $L^2((0, T) \times \Omega)$ for all $T > 0$. Since $r \mapsto r^{\frac{1}{m}}$ is $\frac{1}{m}$ -Hölder continuous, it is easy to check that the previous compactness property implies that (u_δ) is compact in $L^{2m}((0, T) \times \Omega)$ for all $T > 0$. There are thus a function $u \in L^{2m}((0, T) \times \Omega)$ for all $T > 0$ and a sequence $(\delta_n)_{n \geq 1}$ such that

$$u_{\delta_n} \longrightarrow u \text{ in } L^{2m}((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0, \quad (41)$$

for all $T > 0$, owing to Lemma 3.9, we may also assume that

$$u_{\delta_n} \xrightarrow{*} u \text{ in } L^\infty((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0. \quad (42)$$

for all $T > 0$. It readily follows from (41) and (42), and Hölder inequality that (39) holds true. Since elliptic regularity ensure that

$$\|\varphi_{\delta_k} - \varphi_{\delta_n}\|_{W^{2,p}} \leq C(p) \|u_{\delta_k} - u_{\delta_n}\|_p,$$

for all $k \geq 1, n \geq 1$, and $p \in (1, \infty)$, a straightforward consequence of (39) is that $(\varphi_{\delta_n})_{n \geq 1}$ is a Cauchy sequence in $L^p((0, T); W^{2,p}(\Omega))$ and thus converges to some function φ in that space. Finally, the nonnegativity of u follows easily from that of u_{δ_n} by (39). \square

Proof of Theorem 2.2 (existence). It remains to identify the equations solved by the limit (u, φ) of $(u_{\delta_n}, \varphi_{\delta_n})_{n \geq 1}$ constructed in Lemma 4.1. To this end we first note that, owing to (39) and the boundedness of $(u_{\delta_n})_n$ and u in $L^\infty((0, T) \times \Omega)$, we have

$$u_{\delta_n}^m \longrightarrow u^m \text{ in } L^p((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0, \quad (43)$$

for all $T > 0$. Since $(\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}})_{n \geq 1}$ and $(\nabla u_{\delta_n}^m)_{n \geq 1}$ are bounded in $L^2((0, T) \times \Omega)$ for all $T > 0$ by Lemma 3.4 with $p = 2$ and $p = m + 1$, we may extract a further subsequence (not relabeled) such that

$$\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} \rightharpoonup \nabla u^{\frac{m+1}{2}} \text{ in } L^2((0, T) \times \Omega), \quad (44)$$

$$\nabla u_{\delta_n}^m \rightharpoonup \nabla u^m \text{ in } L^2((0, T) \times \Omega), \quad (45)$$

for all $T > 0$. Then if $\psi \in L^4((0, T) \times \Omega; \mathbb{R}^N)$,

$$\begin{aligned} & \left| \int_0^T \int_\Omega \psi \cdot [\nabla(u_{\delta_n} + \delta_n)^m - \nabla u^m] \, dx ds \right| \\ &= \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \left[(u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - u^{\frac{m-1}{2}} \nabla u^{\frac{m+1}{2}} \right] \, dx ds \right| \\ &\leq \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} ((u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} - u^{\frac{m-1}{2}}) \, dx ds \right| \\ &\quad + \frac{2}{m+1} \left| \int_0^T \int_\Omega u^{\frac{m-1}{2}} \psi \cdot \left(\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - \nabla u^{\frac{m+1}{2}} \right) \, dx ds \right| \\ &\leq \frac{2}{m+1} \|\psi\|_4 \|\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}}\|_2 \|(u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} - u^{\frac{m-1}{2}}\|_4 \\ &\quad + \frac{2}{m+1} \left| \int_0^T \int_\Omega u^{\frac{m-1}{2}} \psi \cdot \left(\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - \nabla u^{\frac{m+1}{2}} \right) \, dx ds \right|. \end{aligned}$$

Since $u^{\frac{m-1}{2}} \psi \in L^2((0, T) \times \Omega)$, we deduce from (39) and (44) that the right-hand side of the above inequality converges to zero as $n \rightarrow \infty$. In other words,

$$\nabla(u_{\delta_n} + \delta_n)^m \rightharpoonup \nabla u^m \text{ in } L^{\frac{4}{3}}((0, T) \times \Omega), \quad (46)$$

for all $T > 0$.

Now, we are going to show that (u, φ) in Lemma 4.1 is the desired weak solution in Theorem 2.2. Let $T > 0$ and $\psi \in C^1([0, T] \times \overline{\Omega})$ with $\psi(T) = 0$. The solution of (7) satisfies

$$\int_0^T \int_{\Omega} [\nabla(u_{\delta_n} + \delta_n)^m \cdot \nabla \psi - u_{\delta_n} \nabla \varphi_{\delta_n} \cdot \nabla \psi - u_{\delta_n} \partial_t \psi] \, dx dt = \int_{\Omega} u_0 \psi(0, x) \, dx, \quad (47)$$

and,

$$\int_0^T \int_{\Omega} [\nabla \varphi_{\delta_n} \cdot \nabla \psi + M \psi - u_{\delta_n} \psi] \, dx dt = 0. \quad (48)$$

From (46) we see that

$$\int_0^T \int_{\Omega} \nabla(u_{\delta_n} + \delta_n)^m \cdot \nabla \psi \, dx dt \longrightarrow \int_0^T \int_{\Omega} \nabla u^m \cdot \nabla \psi \, dx dt \text{ as } \delta_n \rightarrow 0.$$

From (39) we get

$$\int_0^T \int_{\Omega} u_{\delta_n} \partial_t \psi \, dx dt \longrightarrow \int_0^T \int_{\Omega} u \partial_t \psi \, dx dt \text{ as } \delta_n \rightarrow 0.$$

From (39) and (40) we get

$$\int_0^T \int_{\Omega} u_{\delta_n} \nabla \varphi_{\delta_n} \cdot \nabla \psi \, dx dt \longrightarrow \int_0^T \int_{\Omega} u \nabla \varphi \cdot \nabla \psi \, dx dt \text{ as } \delta_n \rightarrow 0.$$

Thus we conclude that u satisfies

$$\int_0^T \int_{\Omega} (\nabla u^m \cdot \nabla \psi - u \nabla \varphi \cdot \nabla \psi - u \cdot \partial_t \psi) \, dx dt = \int_{\Omega} u_0(x) \cdot \psi(0, x) \, dx.$$

Similarly, from (40) we see that

$$\int_0^T \int_{\Omega} \nabla \varphi_{\delta_n} \cdot \nabla \psi \, dx dt \longrightarrow \int_0^T \int_{\Omega} \nabla \varphi \cdot \nabla \psi \, dx dt \text{ as } \delta_n \rightarrow 0,$$

and from (39) we see that

$$\int_0^T \int_{\Omega} u_{\delta_n} \psi \, dx dt \longrightarrow \int_0^T \int_{\Omega} u \psi \, dx dt \text{ as } \delta_n \rightarrow 0.$$

Thus, we have constructed a weak solution (u, φ) of (KS). □

4.2 Uniqueness

In this section, we prove the uniqueness statement of Theorem 2.2 under the additionnal assumption (6) on φ . The proof relies on a classical duality technique, and on the method presented in [2]

Proof. The proof estimates the difference of weak solutions in dual space $H^1(\Omega)'$ of $H^1(\Omega)$, motivated by the fact that the nonlinear diffusion is monotone in this norm.

Assume that we have two different weak solutions (u_1, φ_1) and (u_2, φ_2) to equations (1) corresponding to the same initial conditions, and fix $T > 0$. We put

$$(u, \varphi) = (u_1 - u_2, \varphi_1 - \varphi_2) \text{ in } [0, T] \times \Omega.$$

Then φ is the strong solution of

$$\begin{aligned} -\Delta \varphi &= u & \text{in } \Omega, \\ \partial_\nu \varphi &= 0 & \text{on } \partial\Omega, \\ \langle \varphi \rangle &= 0. \end{aligned} \quad (49)$$

Since $\partial_t u \in L^2((0, T); H^1(\Omega)')$, we have

$$-\Delta \partial_t \varphi = \partial_t u_1 - \partial_t u_2 = \partial_t u \text{ in } H^1(\Omega)',$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|_2^2 &= \int_{\Omega} \nabla \varphi \cdot \nabla \partial_t \varphi \, dx \\ &= - \langle \Delta \partial_t \varphi, \varphi \rangle_{(H^1)', H^1} = \langle \partial_t u, \varphi \rangle_{(H^1)', H^1}. \end{aligned} \quad (50)$$

Now it follows from (1) that u satisfies the equation

$$\begin{cases} \partial_t u &= \operatorname{div}(\nabla(u_1^m - u_2^m)) - \operatorname{div}(u_1 \nabla \varphi + u \nabla \varphi_2) \\ \partial_\nu u &= 0 \\ u(0, x) &= 0. \end{cases} \quad (51)$$

Substituting (51) in (50), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|_2^2 = \int_{\Omega} (u_1^m - u_2^m) \Delta \varphi \, dx + \int_{\Omega} u_1 |\nabla \varphi|^2 \, dx + \int_{\Omega} u \nabla \varphi_2 \cdot \nabla \varphi \, dx. \quad (52)$$

The first integral on the right-hand side of (52) is nonnegative due to the fact that $z \mapsto z^m$ is an increasing function. The second integral on the right-hand side of (52) can be estimated by

$$\left| \int_{\Omega} u_1 |\nabla \varphi|^2 \, dx \right| \leq \|u_1\|_{\infty} \int_{\Omega} |\nabla \varphi|^2 \, dx.$$

For the last integral, using an integration by parts we obtain

$$\begin{aligned} \int_{\Omega} u \nabla \varphi_2 \cdot \nabla \varphi \, dx &= - \int_{\Omega} \Delta \varphi \nabla \varphi_2 \cdot \nabla \varphi \, dx \\ &= \int_{\Omega} \nabla \varphi \cdot \nabla (\nabla \varphi_2 \cdot \nabla \varphi) \, dx \\ &= \sum_{i,j} \int_{\Omega} \partial_i \varphi \partial_{ij}^2 \varphi_2 \partial_j \varphi \, dx + \sum_{i,j} \int_{\Omega} \partial_i \varphi \partial_j \varphi_2 \partial_{ij}^2 \varphi \, dx. \end{aligned} \quad (53)$$

integrating by parts the second integral on the right-hand side of (53),

$$\begin{aligned} \sum_{i,j} \int_{\Omega} \partial_i \varphi \partial_j \varphi_2 \partial_{ij}^2 \varphi \, dx &= \sum_{i,j} \frac{1}{2} \int_{\Omega} \partial_j \varphi_2 \partial_j |\partial_i \varphi|^2 \, dx \\ &= \frac{1}{2} \int_{\Omega} \nabla \varphi_2 \cdot \nabla (|\nabla \varphi|^2) \, dx \\ &= -\frac{1}{2} \int_{\Omega} \Delta \varphi_2 |\nabla \varphi|^2 \, dx \\ &\leq C(T) \|\nabla \varphi\|_2^2, \end{aligned}$$

since $-\Delta\varphi_2 = u_2 - \langle u_2 \rangle \in L^\infty((0, T) \times \Omega)$. Together with (53) the previous inequality implies

$$\begin{aligned} \left| \int_{\Omega} u \nabla\varphi_2 \cdot \nabla\varphi \, dx \right| &\leq C(T) \int_{\Omega} (|D^2\varphi_2| + 1) |\nabla\varphi|^2 \, dx. \\ &\leq C(T) (\|\varphi_2\|_{L^\infty((0, T); W^{2, \infty}(\Omega))} + 1) \int_{\Omega} |\nabla\varphi|^2 \, dx, \end{aligned}$$

provided that the $L^\infty((0, T); W^{2, \infty}(\Omega))$ norm of the function φ_2 is bounded. Thus, substituting the above estimates in (52), one finally obtains

$$\frac{d}{dt} \int_{\Omega} |\nabla\varphi|^2 \, dx \leq C(T) \int_{\Omega} |\nabla\varphi|^2 \, dx. \quad (54)$$

Notice that $\|\nabla\varphi(0)\|_2 = 0$ which follows from (49) and the property $u(0) = 0$. Thus, inequality (54) implies

$$\|\nabla\varphi(t)\|_2^2 \leq e^{C(T)t} \|\nabla\varphi(0)\|_2^2 = 0.$$

Consequently, $\nabla\varphi(t) = 0$ for all $t \in [0, T]$ and, since $\langle \varphi(t) \rangle = 0$, we have $\varphi(t) = 0$ for all $t \in [0, T]$. Using (49), we conclude that $u(t) = 0$ for all $t \in [0, T]$. Consequently $(u_1, \varphi_1) = (u_2, \varphi_2)$. \square

ACKNOWLEDGMENT

I thank professors Philippe Laurençot and Marjolaine Puel for their helpful advices and comments during this work.

References

- [1] N. D. Alikakos. L^p bounds of solutions of reaction-diffusion equations. *Communication in Partial Differential Equations* 4(1979), no. 8, 827-868.
- [2] J. Bedrossian, N. Rodriguez and A. Bertozzi. Local global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion, *Nonlinearity* 24 (2011) 1683-1714.
- [3] J. Bedrossian, I. C. Kim. Global Existence and Finite Time Blow-Up for Critical Patlak-Keller-Segel Models with Inhomogeneous Diffusion, preprint arXiv:1108.5301.
- [4] A. Blanchet. On the parabolic-elliptic Patlak-Keller-Segel system in dimension 2 and higher. *To appear in Sémin. Équ. Dériv. Partielles*.
- [5] A. Blanchet. J. A. Carrillo. Ph. Laurençot. Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. *Calc. Var. Partial Differential Equations* 35 (2009), no. 2, 133-168.
- [6] V. Calvez, J. A. Carrillo. Volume effect in the Keller-Segel model: energy estimates preventing blow-up. *J. Math. Pures. Appl.* 86 (2006) 155-175.
- [7] T. Cieřlak, Ph. Laurençot. Finite time blow-up for radially symmetric solutions to a critical quasilinear Smoluchowski-Poisson system. *C.R. Acad. Sci. Paris, ser. I* 347(2009) 237-242.

- [8] T. Cieřlak, M. Winkler. Finite-time blow-up in a quasilinear system of chemotaxis. *Nonlinearity* 21 (2008) 1057-1076.
- [9] J. Dolbeault, B. Perthame. Optimal critical mass in the two-dimensional Keller-Segel model in \mathbb{R}^2 . *C. R. Math. Acad. Sci. Paris* 339 (2004), no. 9, 611-616.
- [10] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.* 105(3):103-165, 2003.
- [11] D. Horstmann. Lyapunov functions and L^p -estimates for a class of reaction-diffusion systems. *Colloq. math.* 87 (2001) no. 1, 113-127.
- [12] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.* 329(2)(1992) 819-824.
- [13] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399-415, 1970.
- [14] R. Kowalczyk. Preventing blow-up in a chemotaxis model. *J. Math. Anal. Appl.* 305 (2005) 566-588.
- [15] T. Nagai. Blow-up of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains. *J. Inequal. Appl.* 6 (2001) 37-55.
- [16] T. Nagai. Blow-up of radially symmetric solutions to a chemotaxis system, *Advances in Mathematical Sciences and Applications* 5 (1995), no. 2, 581-601.
- [17] C.S. Patlak, Random walk with persistence and external bias, *Bull. Math. Biophys.* 15 (1953) 311-338.
- [18] B. Perthame, PDE models for chemotactic movements. Parabolic, hyperbolic and kinetic, *Appl. Math.* 49 (6) (2005) 539-564.
- [19] J. Simon, Compact sets in the space $L^p(0, T; B)$. 1987, *Annali di Matematica Pura ed Applicata (IV)*, vol. CXLVI, 65-96.
- [20] Y. Sugiyama. Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems. *Differential and Integral Equations* 19 (2006), no. 8, 841-876.
- [21] Y. Sugiyama. Time global existence and asymptotic behavior for solutions to degenerate quasi-linear parabolic systems of chemotaxis. *Differential and Integral Equations* 20 (2007), no. 2, 133-180.